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# On the well-posedness of some mechanical variational problems

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SUMMARY

The paper aims to establish the existence and uniqueness of the solution of some variational problems constituting the basis of finite element modellings encountered in mechanics and civil engineering. And indeed, by expanding to the approximate problems coming from the space discretization, such theoretical results contribute to strengthen the robustness of the modelling softwares and the quality of their numerical results. More particularly, three kinds of mixed variational problems involving rheological non-linearities are considered here : the evolution problems of incompressible continua (solids or fluids) subjected to quasistatic small transformations, the problems of hydromechanical coupling and those coming from quasistatic large transformations of continua. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: Mixed variational formulation; existence and uniqueness; finite elements

## 1. STATEMENT OF THE PROBLEM

Among the large field that constitute the reflections concerning the quality of finite element computations in mechanics and civil engineering, the topics relating to the existence and uniqueness of the solutions of the variational problems constituting the basis of this method are of great importance. And indeed, by expanding to the approximate problems coming from the space discretization, such theoretical results contribute to strengthen the robustness of the modelling softwares and the quality of their numerical results.

So, in this paper we purpose to establish some existence and uniqueness results for the solution of variational problems constituting the basis of finite element approximations in mechanics.

The problems considered take the form:

$$(P)\left\{\begin{array}{lllll} \text{Find } (\boldsymbol{\lambda},\boldsymbol{\mu}) \in V\times M \text{ such that} & & & & \\ a(\boldsymbol{\lambda},\mathbf{u}) & - \; b(\mathbf{u},\boldsymbol{\mu}) & = & l(\mathbf{u}) & \forall \mathbf{u}\in V \\ b(\boldsymbol{\lambda},\mathbf{v}) & + \; c(\boldsymbol{\mu},\mathbf{v}) & = & h(\mathbf{v}) & \forall \mathbf{v}\in M \end{array}\right.$$

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where $V$ and $M$ are two Hilbert spaces with dual spaces $V'$ and $M'$, respectively, $a : V \times V \mapsto \mathrm{I\!R}$ a form linear regarding its second argument but non-linear with respect to the first one (this non-linearity arises from the rheological features of the continuum), $b : V \times M \mapsto \mathrm{I\!R}$ and $c : M \times M \mapsto \mathrm{I\!R}$ two bilinear and continuous forms with continuity constants $c_b$ and $c_c$, respectively, whereas $l \in V'$ and $h \in M'$.

## 2. NOTATIONS — PRELIMINARY RESULT

Let $E$ be a given Hilbert space (for instance $V$ or $M$) with dual space $E'$. We denote by $(.,.)_E$ the inner product of $E$, by $\|.\|_E$ the corresponding norm and by $\|.\|_{E'}$ the dual norm. Given $g \in E'$, we denote by $\mathbf{g}$ the unique element of $E$ such that $(\mathbf{g}, \mathbf{w})_E = g(\mathbf{w})$ $\forall \mathbf{w} \in E$ (Riesz's representation theorem). Moreover, we have $\|\mathbf{g}\|_E = \|g\|_{E'}$.

Since the form $a$ is linear and continuous with respect to its second argument, there exists a unique operator $A : V \mapsto V'$ such that we have, $\forall \boldsymbol{\lambda} \in V$, $A\boldsymbol{\lambda}(\mathbf{u}) = a(\boldsymbol{\lambda}, \mathbf{u})$ $\forall \mathbf{u} \in V$. So, we denote by $\mathbf{A} : V \mapsto V$ the operator (non-linear like $A$) which associates to any $\boldsymbol{\lambda} \in V$ the unique element $\mathbf{A}\boldsymbol{\lambda} \in V$ linked to $A\boldsymbol{\lambda}$ by the Riesz's representation theorem.

Otherwise, we introduce the linear and continuous operators $B \in \mathcal{L}(V; M')$, ${}^{\mathrm{t}}B \in \mathcal{L}(M; V')$ and $C \in \mathcal{L}(M; M')$ respectively defined by

$$\begin{array}{lcrcll} \forall \mathbf{u} \in V & : & B\mathbf{u}(\boldsymbol{\mu}) & = & b(\mathbf{u}, \boldsymbol{\mu}) & \forall \boldsymbol{\mu} \in M \\ \forall \mathbf{v} \in M & : & {}^{\mathrm{t}}B\mathbf{v}(\boldsymbol{\lambda}) & = & b(\boldsymbol{\lambda}, \mathbf{v}) & \forall \boldsymbol{\lambda} \in V \\ \forall \boldsymbol{\mu} \in M & : & C\boldsymbol{\mu}(\mathbf{v}) & = & c(\boldsymbol{\mu}, \mathbf{v}) & \forall \mathbf{v} \in M \end{array}$$

from which we build, by combining them (as described above for $A$) with the appropriate one-to-one mapping of representation, the linear and continuous operators $\mathbf{B} \in \mathcal{L}(V; M)$, ${}^{\mathrm{t}}\mathbf{B} \in \mathcal{L}(M; V)$ and $\mathbf{C} \in \mathcal{L}(M; M)$.

Let now $K$ be the kernel of $B$ (or, that is the same, of $\mathbf{B}$), $K^{\perp}$ the orthogonal set of $K$ in $V$ and $K^{\circ}$ the polar set of $K$ in $V'$.

We have then the

**Lemma 1 ([9])** *The three following propositions are equivalent.*

1. *There exists* $\beta \in \mathrm{I\!R}^{+\star}$ *such that* $\displaystyle\inf_{\boldsymbol{\mu} \in M^\star} \sup_{\boldsymbol{\lambda} \in V^\star} \frac{b(\boldsymbol{\lambda}, \boldsymbol{\mu})}{\|\boldsymbol{\lambda}\|_V \|\boldsymbol{\mu}\|_M} \geq \beta$.
2. *The operator* $B$ *is an isomorphism from* $K^{\perp}$ *onto* $M'$. *Moreover, we have* $\|B\boldsymbol{\lambda}\|_{M'} \geq \beta \|\boldsymbol{\lambda}\|_V$, $\forall \boldsymbol{\lambda} \in K^{\perp}$.
3. *The operator* ${}^{\mathrm{t}}B$ *is an isomorphism from* $M$ *onto* $K^{\circ}$. *Moreover, we have* $\|{}^{\mathrm{t}}B\boldsymbol{\mu}\|_{V'} \geq \beta \|\boldsymbol{\mu}\|_M$, $\forall \boldsymbol{\mu} \in M$.

The inf-sup condition in the proposition 1 of lemma 1 is also known as LBB condition, where the L in acronym LBB stands for Ladyzhenskaya [10], the first B for Babuška [1] and the second one for Brezzi [2, 3].

Eventually, we conclude this section by establishing the

**Theorem 1.** *Let* $E$ *be a Hilbert space,* $f : E \times E \mapsto \mathrm{I\!R}$ *and* $K$ *a closed subspace of* $E$. *Assume that the form* $f$ *is linear and continuous with respect to its second argument, i.e. that there exists an operator* $\mathbf{F} : E \mapsto E$ *such that* $f(\boldsymbol{\eta}, \mathbf{w}) = (\mathbf{F}\boldsymbol{\eta}, \mathbf{w})_E$ $\forall (\boldsymbol{\eta}, \mathbf{w}) \in E \times E$. *Assume in addition that* $\mathbf{F}$ *satisfies the two following properties*

*1.* $\exists c_f \in \mathbb{R}^{+\star} : \|\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}'\|_E \leq c_f \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E \quad \forall(\boldsymbol{\eta}, \boldsymbol{\eta}') \in K \times K$

*2.* $\exists \alpha_f \in \mathbb{R}^{+\star} : (\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}', \boldsymbol{\eta} - \boldsymbol{\eta}')_E \geq \alpha_f \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E^2 \quad \forall(\boldsymbol{\eta}, \boldsymbol{\eta}') \in K \times K$

*Then,* $\forall g \in E'$, *the problem*

$$(P_0) \left\{ \begin{array}{l} \text{Find } \boldsymbol{\eta} \in K \text{ such that} \\ \quad f(\boldsymbol{\eta}, \mathbf{w}) \;\; = \;\; g(\mathbf{w}) \qquad \forall \mathbf{w} \in K \end{array} \right.$$

*has a unique solution* $\boldsymbol{\eta} \in K$.

**Proof of theorem 1.** Let $\rho$ be a strictly positive constant. We have

$$\begin{array}{rcl} f(\boldsymbol{\eta}, \mathbf{w}) = g(\mathbf{w}) \quad \forall \mathbf{w} \in K & \Leftrightarrow & (\mathbf{F}\boldsymbol{\eta}, \mathbf{w})_E = (\mathbf{g}, \mathbf{w})_E \quad \forall \mathbf{w} \in K \\ & \Leftrightarrow & (\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta} + \boldsymbol{\eta} - \boldsymbol{\eta}, \mathbf{w})_E = 0 \quad \forall \mathbf{w} \in K \\ & \Leftrightarrow & \boldsymbol{\eta} = \mathbf{P}_K(\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta} + \boldsymbol{\eta}) \end{array}$$

where $\mathbf{P}_K$ denotes the operator of orthogonal projection onto $K$. So, let us set, $\forall \boldsymbol{\eta} \in K$, $\mathbf{S}\boldsymbol{\eta} = \mathbf{P}_K(\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta} + \boldsymbol{\eta})$. The idea is to adjust $\rho$ in order that $\mathbf{S}$ be a strict contraction.

Let us notice that $\forall(\boldsymbol{\eta}, \boldsymbol{\eta}') \in K \times K$ we have

$$\begin{array}{rcl} \|\mathbf{S}\boldsymbol{\eta} - \mathbf{S}\boldsymbol{\eta}'\|_E^2 & = & \|\mathbf{P}_K(\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta} + \boldsymbol{\eta}) - \mathbf{P}_K(\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta}' + \boldsymbol{\eta}')\|_E^2 \\ & \leq & \|(\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta} + \boldsymbol{\eta}) - (\rho\mathbf{g} - \rho\mathbf{F}\boldsymbol{\eta}' + \boldsymbol{\eta}')\|_E^2 \\ & = & \|\boldsymbol{\eta} - \boldsymbol{\eta}' - \rho(\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}')\|_E^2 \\ & = & \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E^2 + \rho^2\|\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}'\|_E^2 - 2\rho(\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}', \boldsymbol{\eta} - \boldsymbol{\eta}')_E \\ & \leq & \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E^2(1 + \rho^2 c_f^2 - 2\rho\alpha_f) \end{array}$$

which gives

$$\|\mathbf{S}\boldsymbol{\eta} - \mathbf{S}\boldsymbol{\eta}'\|_E \leq k\|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E \quad \forall(\boldsymbol{\eta}, \boldsymbol{\eta}') \in K \times K$$

with $k = \sqrt{1 + \rho^2 c_f^2 - 2\rho\alpha_f}$. Therefore, if $\rho \in ]0, 2\frac{\alpha_f}{c_f^2}[$ then $k \in ]0, 1[$ and $\mathbf{S}$ is a strict contraction, which establishes the theorem 1.

Note that theorem 1 can also be proved by using the Minty-Browder theorem [12, 4] which is based on the properties of continuity, monotony and coercivity of the operator $\mathbf{F}$. And indeed, the condition 1 of theorem 1 involves the continuity of $\mathbf{F}$ whereas condition 2 entails both monotony and coercivity.

## 3. EXISTENCE AND UNIQUENESS RESULTS

### *3.1. First class of problems :* $c = 0_{\mathcal{L}(M \times M;\mathbb{R})}$

This first case, with $c = 0_{\mathcal{L}(M \times M;\mathbb{R})}$, corresponds to the set of mechanical problems involving incompressible continua (solids or fluids) subjected to quasistatic small transformations [9, 14]. For such problems, $\boldsymbol{\lambda}$ is the unknown field of velocity (or displacement) and $\boldsymbol{\mu}$ is the one of pressure. Otherwise, we have also $c = 0_{\mathcal{L}(M \times M;\mathbb{R})}$ when a two fields mixed variational formulation is used for modelling quasistatic small transformations of compressible continua [11]. In that last case $\boldsymbol{\lambda}$ is the field of the Cauchy's stresses whereas $\boldsymbol{\mu}$ is the one of displacement.

Let us now establish the

**Theorem 2.** *Assume that the bilinear form $b$ satisfies the inf-sup condition of lemma 1. Assume in addition that $\forall(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_0, \boldsymbol{\lambda}'_0) \in K^\perp \times K \times K$ the operator $\mathbf{A}$ satisfies the two following properties*

1. $\exists c_a \in \mathbb{R}^{+\star} : \|\mathbf{A}[\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_0] - \mathbf{A}[\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}'_0]\|_V \leq c_a \|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|_V$
2. $\exists \alpha_a \in \mathbb{R}^{+\star} : (\mathbf{A}[\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_0] - \mathbf{A}[\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}'_0], \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0)_V \geq \alpha_a \|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|^2_V$

*Then, $\forall(l, h) \in V' \times M'$, the problem*

$$(P) \left\{ \begin{array}{lcccl} \text{Find } (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M \text{ such that} & & & & \\ a(\boldsymbol{\lambda}, \mathbf{u}) \quad - \quad b(\mathbf{u}, \boldsymbol{\mu}) & = & l(\mathbf{u}) & & \forall \mathbf{u} \in V \\ b(\boldsymbol{\lambda}, \mathbf{v}) & = & h(\mathbf{v}) & & \forall \mathbf{v} \in M \end{array} \right.$$

*has a unique solution $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M$.*

**Proof of theorem 2.** We first notice that $b(\boldsymbol{\lambda}, \mathbf{v}) = h(\mathbf{v})\ \forall \mathbf{v} \in M \Leftrightarrow B\boldsymbol{\lambda} = h$. From the second proposition of lemma 1, there exists a unique $\boldsymbol{\lambda}_1 \in K^\perp$ such that $B\boldsymbol{\lambda}_1 = h$. Thus, $B\boldsymbol{\lambda} = h \Leftrightarrow \boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1$ with $\boldsymbol{\lambda}_0 \in K$, so that $(P)$ is equivalent to the problem

$$(P_{11}) \left\{ \begin{array}{lcccl} \text{Find } (\boldsymbol{\lambda}_0, \boldsymbol{\mu}) \in K \times M \text{ such that} & & & & \\ a(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1, \mathbf{u}) \quad - \quad b(\mathbf{u}, \boldsymbol{\mu}) & = & l(\mathbf{u}) & & \forall \mathbf{u} \in V \end{array} \right.$$

Now, if $(P_{11})$ has a solution then $\boldsymbol{\lambda}_0$ is necessarily a solution of

$$(P_{12}) \left\{ \begin{array}{lcccl} \text{Find } \boldsymbol{\lambda}_0 \in K \text{ such that} & & & & \\ a(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1, \mathbf{u}) & = & l(\mathbf{u}) & & \forall \mathbf{u} \in K \end{array} \right.$$

So, let us set, $\forall(\boldsymbol{\lambda}_0, \mathbf{u}) \in K \times K$, $f(\boldsymbol{\lambda}_0, \mathbf{u}) = a(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1, \mathbf{u})$. We obtain then, $\forall \boldsymbol{\lambda}_0 \in K$, $f(\boldsymbol{\lambda}_0, \mathbf{u}) = (\mathbf{F}\boldsymbol{\lambda}_0, \mathbf{u})_V\ \ \forall \mathbf{u} \in V$ with $\mathbf{F}\boldsymbol{\lambda}_0 = \mathbf{A}[\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1]$. Moreover, the properties 1 and 2 satisfied by the operator $\mathbf{A}$ show that we have

$$\begin{array}{l} \|\mathbf{F}\boldsymbol{\lambda}_0 - \mathbf{F}\boldsymbol{\lambda}'_0\|_V \leq c_a \|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|_V \quad \forall(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}'_0) \in K \times K \\ (\mathbf{F}\boldsymbol{\lambda}_0 - \mathbf{F}\boldsymbol{\lambda}'_0, \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0)_V \geq \alpha_a \|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|^2_V \quad \forall(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}'_0) \in K \times K \end{array}$$

Hence, it follows from theorem 1 that $(P_{12})$ has a unique solution $\boldsymbol{\lambda}_0 \in K$.

Moreover, $a(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1, \mathbf{u}) = l(\mathbf{u})\ \forall \mathbf{u} \in K \Leftrightarrow A[\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1] - l \in K^\circ$. Thus, the existence and uniqueness of $\boldsymbol{\mu}$ ensues from the third proposition of lemma 1 since the solution $(\boldsymbol{\lambda}_0, \boldsymbol{\mu})$ of $(P_{11})$ satisfies the relation ${}^{\mathrm{t}}B\boldsymbol{\mu} = A[\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1] - l$. Therefore, $(\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1, \boldsymbol{\mu})$ is the unique solution of $(P)$, which ends the proof of theorem 2.

### *3.2. Second class of problems : c is coercive*

The second case deals with hydromechanical coupling problems such as consolidation of clays, for which quasistatic small transformations are again assumed. So, $\boldsymbol{\lambda}$ is the displacement (or velocity) field, $\boldsymbol{\mu}$ the one of pore pressure and the (nonzero) bilinear form $c$ is now continuous and coercive [8].

We have then the

**Theorem 3.** *Assume that the continuous and bilinear form $c$ is coercive, i.e. that there exists a strictly positive constant $\alpha_c$ such that $c(\mathbf{v}, \mathbf{v}) \geq \alpha_c \|\mathbf{v}\|^2_M\ \forall \mathbf{v} \in M$. Assume in addition that the operator $\mathbf{A}$ satisfies the two following properties*

1. $\exists c_a \in \mathbb{R}^{+\star} : \|\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}'\|_V \leq c_a \|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|_V \quad \forall(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \in V \times V$
2. $\exists \alpha_a \in \mathbb{R}^{+\star} : (\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}', \boldsymbol{\lambda} - \boldsymbol{\lambda}')_V \geq \alpha_a \|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|_V^2 \quad \forall(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \in V \times V$

*Then,* $\forall(l, h) \in V' \times M'$*, the problem*

$$(P) \left\{ \begin{array}{l} \text{Find } (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M \text{ such that} \\ \begin{array}{ccccccc} a(\boldsymbol{\lambda}, \mathbf{u}) & - & b(\mathbf{u}, \boldsymbol{\mu}) & = & l(\mathbf{u}) & & \forall \mathbf{u} \in V \\ b(\boldsymbol{\lambda}, \mathbf{v}) & + & c(\boldsymbol{\mu}, \mathbf{v}) & = & h(\mathbf{v}) & & \forall \mathbf{v} \in M \end{array} \end{array} \right.$$

*has a unique solution* $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M$*.*

**Proof of theorem 3.** To begin with we set $E = V \times M$, $\boldsymbol{\eta} = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in E$ and $\mathbf{w} = (\mathbf{u}, \mathbf{v}) \in E$. So, $E$ is a Hilbert space when equipped with the inner product $(\boldsymbol{\eta}, \mathbf{w})_E = (\boldsymbol{\lambda}, \mathbf{u})_V + (\boldsymbol{\mu}, \mathbf{v})_M$. Otherwise, we introduce the forms $f : E \times E \mapsto \mathbb{R}$ and $g : E \mapsto \mathbb{R}$ defined by

$$\begin{array}{rcl} f(\boldsymbol{\eta}, \mathbf{w}) & = & a(\boldsymbol{\lambda}, \mathbf{u}) - b(\mathbf{u}, \boldsymbol{\mu}) + b(\boldsymbol{\lambda}, \mathbf{v}) + c(\boldsymbol{\mu}, \mathbf{v}) \quad \forall(\boldsymbol{\eta}, \mathbf{w}) \in E \times E \\ g(\mathbf{w}) & = & l(\mathbf{u}) + h(\mathbf{v}) \quad \forall \mathbf{w} \in E \end{array}$$

Then, the problem $(P)$ is equivalent to

$$(P_2) \left\{ \begin{array}{l} \text{Find } \boldsymbol{\eta} \in E \text{ such that} \\ f(\boldsymbol{\eta}, \mathbf{w}) \quad = \quad g(\mathbf{w}) \qquad \forall \mathbf{w} \in E \end{array} \right.$$

We first notice that $(l, h) \in V' \times M' \Rightarrow g \in E'$. And indeed, let $c_l$ and $c_h$ be the continuity constants of $l$ and $h$, respectively. We get then, $\forall \mathbf{w} = (\mathbf{u}, \mathbf{v}) \in E$,

$$|g(\mathbf{w})| \leq |l(\mathbf{u})| + |h(\mathbf{v})| \leq c_l \|\mathbf{u}\|_V + c_h \|\mathbf{v}\|_M \leq (c_l + c_h)\|\mathbf{w}\|_E$$

Otherwise we have, $\forall(\boldsymbol{\eta}, \mathbf{w}) \in E \times E$,

$$\begin{array}{rcl} f(\boldsymbol{\eta}, \mathbf{w}) & = & a(\boldsymbol{\lambda}, \mathbf{u}) - b(\mathbf{u}, \boldsymbol{\mu}) + b(\boldsymbol{\lambda}, \mathbf{v}) + c(\boldsymbol{\mu}, \mathbf{v}) \\ & = & (\mathbf{A}\boldsymbol{\lambda}, \mathbf{u})_V - ({}^{\mathrm{t}}\mathbf{B}\boldsymbol{\mu}, \mathbf{u})_V + (\mathbf{B}\boldsymbol{\lambda}, \mathbf{v})_M + (\mathbf{C}\boldsymbol{\mu}, \mathbf{v})_M \\ & = & (\mathbf{A}\boldsymbol{\lambda} - {}^{\mathrm{t}}\mathbf{B}\boldsymbol{\mu}, \mathbf{u})_V + (\mathbf{B}\boldsymbol{\lambda} + \mathbf{C}\boldsymbol{\mu}, \mathbf{v})_M \end{array}$$

which shows that $f(\boldsymbol{\eta}, \mathbf{w}) = (\mathbf{F}\boldsymbol{\eta}, \mathbf{w})_E \;\; \forall(\boldsymbol{\eta}, \mathbf{w}) \in E \times E$, where $\mathbf{F} : E \mapsto E$ is defined by $\mathbf{F}\boldsymbol{\eta} = (\mathbf{A}\boldsymbol{\lambda} - {}^{\mathrm{t}}\mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\lambda} + \mathbf{C}\boldsymbol{\mu}) \;\forall \boldsymbol{\eta} = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in E$. So, it remains only for us to prove that the operator $\mathbf{F}$ satisfies all requirements of theorem 1.

For that purpose let $\boldsymbol{\eta} = (\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $\boldsymbol{\eta}' = (\boldsymbol{\lambda}', \boldsymbol{\mu}')$ be any given elements of $E$. We have

$$\begin{array}{rcl} \mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}' & = & \big((\mathbf{A}\boldsymbol{\lambda} - {}^{\mathrm{t}}\mathbf{B}\boldsymbol{\mu}) - (\mathbf{A}\boldsymbol{\lambda}' - {}^{\mathrm{t}}\mathbf{B}\boldsymbol{\mu}'), (\mathbf{B}\boldsymbol{\lambda} + \mathbf{C}\boldsymbol{\mu}) - (\mathbf{B}\boldsymbol{\lambda}' + \mathbf{C}\boldsymbol{\mu}')\big) \\ & = & \big(\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}' - {}^{\mathrm{t}}\mathbf{B}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \mathbf{B}[\boldsymbol{\lambda} - \boldsymbol{\lambda}'] + \mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}']\big) \end{array}$$

Therefore

$$\|\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}'\|_E^2 = \big\|\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}' - {}^{\mathrm{t}}\mathbf{B}[\boldsymbol{\mu} - \boldsymbol{\mu}']\big\|_V^2 + \big\|\mathbf{B}[\boldsymbol{\lambda} - \boldsymbol{\lambda}'] + \mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}']\big\|_M^2$$

which gives

$$\begin{array}{rcl} \|\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}'\|_E & \leq & \big\|\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}' - {}^{\mathrm{t}}\mathbf{B}[\boldsymbol{\mu} - \boldsymbol{\mu}']\big\|_V + \big\|\mathbf{B}[\boldsymbol{\lambda} - \boldsymbol{\lambda}'] + \mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}']\big\|_M \\ & \leq & \big\|\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}'\big\|_V + \|{}^{\mathrm{t}}\mathbf{B}[\boldsymbol{\mu} - \boldsymbol{\mu}']\|_V + \big\|\mathbf{B}[\boldsymbol{\lambda} - \boldsymbol{\lambda}']\big\|_M + \|\mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}']\|_M \\ & \leq & c_a \big\|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\big\|_V + c_b \|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_M + c_b \big\|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\big\|_V + c_c \|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_M \\ & \leq & (c_a + 2c_b + c_c) \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E \end{array}$$

Hence, $\|\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}'\|_E \leq c_f \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E$ with $c_f = c_a + 2c_b + c_c$.

Otherwise, it follows from the coercivity of $c$ that $(\mathbf{C}\mathbf{v}, \mathbf{v})_M \geq \alpha_c \|\mathbf{v}\|_M^2$, $\forall \mathbf{v} \in M$. We obtain then

$$
\begin{aligned}
(\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}', \boldsymbol{\eta} - \boldsymbol{\eta}')_E &= \big(\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}' - {}^{\mathrm{t}}\mathbf{B}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \boldsymbol{\lambda} - \boldsymbol{\lambda}'\big)_V \\
&\quad + \big(\mathbf{B}[\boldsymbol{\lambda} - \boldsymbol{\lambda}'] + \mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \boldsymbol{\mu} - \boldsymbol{\mu}'\big)_M \\
&= \big(\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}', \boldsymbol{\lambda} - \boldsymbol{\lambda}'\big)_V + (\mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \boldsymbol{\mu} - \boldsymbol{\mu}')_M \\
&\quad - \big({}^{\mathrm{t}}\mathbf{B}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \boldsymbol{\lambda} - \boldsymbol{\lambda}'\big)_V + \big(\mathbf{B}[\boldsymbol{\lambda} - \boldsymbol{\lambda}'], \boldsymbol{\mu} - \boldsymbol{\mu}'\big)_M \\
&= \big(\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}', \boldsymbol{\lambda} - \boldsymbol{\lambda}'\big)_V + (\mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \boldsymbol{\mu} - \boldsymbol{\mu}')_M \\
&\quad - b(\boldsymbol{\lambda} - \boldsymbol{\lambda}', \boldsymbol{\mu} - \boldsymbol{\mu}') + b(\boldsymbol{\lambda} - \boldsymbol{\lambda}', \boldsymbol{\mu} - \boldsymbol{\mu}') \\
&= \big(\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}', \boldsymbol{\lambda} - \boldsymbol{\lambda}'\big)_V + (\mathbf{C}[\boldsymbol{\mu} - \boldsymbol{\mu}'], \boldsymbol{\mu} - \boldsymbol{\mu}')_M \\
&\geq \alpha_a \|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|_V^2 + \alpha_c \|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_M^2
\end{aligned}
$$

Thus, $(\mathbf{F}\boldsymbol{\eta} - \mathbf{F}\boldsymbol{\eta}', \boldsymbol{\eta} - \boldsymbol{\eta}')_E \geq \alpha_f \|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_E^2$ with $\alpha_f = \min\{\alpha_a, \alpha_c\} > 0$.

Hence, all requirements of theorem 1 hold and the theorem 3 is established.

### *3.3. Third class of problems : c is non coercive*

Eventually, for large transformations (i.e. large displacements, large strains and large rotations) of elastoviscoplastic continua, $\boldsymbol{\lambda}$ is the unknown field of objective stress rate and $\boldsymbol{\mu}$ the velocity one [14, 6, 15]. The bilinear form $c$ remains continuous but turns non-coercive, so that we have now the

**Theorem 4.** *Assume that the bilinear form $b$ satisfies the inf-sup condition of lemma 1. Assume in addition that the operator $\mathbf{A}$ satisfies, $\forall(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \in V \times V$ and $\forall(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_0, \boldsymbol{\lambda}_0') \in K^\perp \times K \times K$, the two following properties*

1. $\exists c_a \in \mathrm{I\!R}^{+\star} : \|\mathbf{A}\boldsymbol{\lambda} - \mathbf{A}\boldsymbol{\lambda}'\|_V \leq c_a \|\boldsymbol{\lambda} - \boldsymbol{\lambda}'\|_V$
2. $\exists \alpha_a \in \mathrm{I\!R}^{+\star} : (\mathbf{A}[\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_0] - \mathbf{A}[\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_0'], \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_0')_V \geq \alpha_a \|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_0'\|_V^2$

*Finally, assume that $\frac{c_a c_c}{\beta^2}(1 + \frac{c_a}{\alpha_a}) < 1$. Then, $\forall(l, h) \in V' \times M'$, the problem*

$$
(P) \left\{ \begin{array}{llllll} \multicolumn{6}{l}{\text{Find } (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M \text{ such that}} \\ a(\boldsymbol{\lambda}, \mathbf{u}) & - & b(\mathbf{u}, \boldsymbol{\mu}) & = & l(\mathbf{u}) & \forall \mathbf{u} \in V \\ b(\boldsymbol{\lambda}, \mathbf{v}) & + & c(\boldsymbol{\mu}, \mathbf{v}) & = & h(\mathbf{v}) & \forall \mathbf{v} \in M \end{array} \right.
$$

*has a unique solution $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M$.*

**Proof of theorem 4.** Let $\boldsymbol{\mu}$ be any given element of $M$ and let us consider the problem

$$
(P_\mu) \left\{ \begin{array}{llllll} \multicolumn{6}{l}{\text{Find } (\mathbf{S}\boldsymbol{\mu}, \mathbf{T}\boldsymbol{\mu}) \in V \times M \text{ such that}} \\ a(\mathbf{S}\boldsymbol{\mu}, \mathbf{u}) & - & b(\mathbf{u}, \mathbf{T}\boldsymbol{\mu}) & = & l(\mathbf{u}) & \forall \mathbf{u} \in V \\ b(\mathbf{S}\boldsymbol{\mu}, \mathbf{v}) & + & c(\boldsymbol{\mu}, \mathbf{v}) & = & h(\mathbf{v}) & \forall \mathbf{v} \in M \end{array} \right.
$$

It is easy to check that problem $(P_\mu)$ has a unique solution $(\mathbf{S}\boldsymbol{\mu}, \mathbf{T}\boldsymbol{\mu}) \in V \times M$. And indeed, we only have to apply theorem 2 after replacing $h$ by $h - C\boldsymbol{\mu} \in M'$ and then $c$ by $0_{\mathcal{L}(M \times M; \mathrm{I\!R})}$. From this it follows immediately that $(P)$ has a unique solution $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in V \times M$ if and only if the mapping $\mathbf{T} : M \mapsto M$ which associates $\mathbf{T}\boldsymbol{\mu}$ with $\boldsymbol{\mu}$ has a unique fixed point. Let us show that this holds as soon as $\frac{c_a c_c}{\beta^2}(1 + \frac{c_a}{\alpha_a}) < 1$.

For that purpose let $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ be any given elements of $M$ and let $(\mathbf{S}\boldsymbol{\mu}, \mathbf{T}\boldsymbol{\mu})$ (resp. $(\mathbf{S}\boldsymbol{\mu}', \mathbf{T}\boldsymbol{\mu}')$) $\in V \times M$ the unique solution of $(P_\mu)$ (resp. $(P_{\mu'})$). Since from the second proposition of lemma 1 the operator $B$ is an isomorphism from $K^\perp$ onto $M'$, there exists a unique $\boldsymbol{\lambda}_1$ (resp. $\boldsymbol{\lambda}'_1$) $\in K^\perp$ such that $B\boldsymbol{\lambda}_1 = h - C\boldsymbol{\mu}$ (resp. $B\boldsymbol{\lambda}'_1 = h - C\boldsymbol{\mu}'$ ). Thus, we have $B[\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1] = -C[\boldsymbol{\mu} - \boldsymbol{\mu}']$ and therefore $\|C[\boldsymbol{\mu} - \boldsymbol{\mu}']\|_{M'} = \|B[\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1]\|_{M'} \geq \beta\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1\|_V$, which gives, taking into account the continuity of the linear operator $C$,

$$\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1\|_V \leq \frac{c_c}{\beta}\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_M \tag{1}$$

We have then $\mathbf{S}\boldsymbol{\mu} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_0$ (resp. $\mathbf{S}\boldsymbol{\mu}' = \boldsymbol{\lambda}'_1 + \boldsymbol{\lambda}'_0$ ) with $\boldsymbol{\lambda}_0$ (resp. $\boldsymbol{\lambda}'_0$) $\in K$. Moreover, $\forall \mathbf{u} \in K$, $a(\mathbf{S}\boldsymbol{\mu}, \mathbf{u}) = a(\mathbf{S}\boldsymbol{\mu}', \mathbf{u}) = l(\mathbf{u})$ which gives, $\forall \mathbf{u} \in K$,

$$\begin{aligned}
& a(\mathbf{S}\boldsymbol{\mu}, \mathbf{u}) = a(\mathbf{S}\boldsymbol{\mu}', \mathbf{u}) \\
\Leftrightarrow\quad & (\mathbf{A}[\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1], \mathbf{u})_V = \big(\mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}'_1], \mathbf{u}\big)_V \\
\Leftrightarrow\quad & \big(\mathbf{A}[\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1] - \mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}_1], \mathbf{u}\big)_V = \big(\mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}'_1] - \mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}_1], \mathbf{u}\big)_V
\end{aligned}$$

So, let us set $\mathbf{u} = \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0$. We get

$$\big(\mathbf{A}[\boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1] - \mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}_1], \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\big)_V = \big(\mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}'_1] - \mathbf{A}[\boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}_1], \boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\big)_V$$

so that we obtain, taking into account the properties of the operator $\mathbf{A}$,

$$\alpha_a\|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|_V^2 \leq c_a\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1\|_V\|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|_V$$

that is to say

$$\|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|_V \leq \frac{c_a}{\alpha_a}\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1\|_V \tag{2}$$

Otherwise, we have $A[\mathbf{S}\boldsymbol{\mu}] - {}^{\mathrm{t}}B[\mathbf{T}\boldsymbol{\mu}] = l$ (resp. $A[\mathbf{S}\boldsymbol{\mu}'] - {}^{\mathrm{t}}B[\mathbf{T}\boldsymbol{\mu}'] = l$ ) and thus $A[\mathbf{S}\boldsymbol{\mu}] - A[\mathbf{S}\boldsymbol{\mu}'] = {}^{\mathrm{t}}B[\mathbf{T}\boldsymbol{\mu} - \mathbf{T}\boldsymbol{\mu}']$, which gives, taking into account the third proposition of lemma 1, $\|A[\mathbf{S}\boldsymbol{\mu}] - A[\mathbf{S}\boldsymbol{\mu}']\|_{V'} \geq \beta\|\mathbf{T}\boldsymbol{\mu} - \mathbf{T}\boldsymbol{\mu}'\|_M$.

We have also $\|A[\mathbf{S}\boldsymbol{\mu}] - A[\mathbf{S}\boldsymbol{\mu}']\|_{V'} = \|\mathbf{A}[\mathbf{S}\boldsymbol{\mu}] - \mathbf{A}[\mathbf{S}\boldsymbol{\mu}']\|_V \leq c_a\|\mathbf{S}\boldsymbol{\mu} - \mathbf{S}\boldsymbol{\mu}'\|_V$ so that we get

$$\|\mathbf{T}\boldsymbol{\mu} - \mathbf{T}\boldsymbol{\mu}'\|_M \leq \frac{c_a}{\beta}\|\mathbf{S}\boldsymbol{\mu} - \mathbf{S}\boldsymbol{\mu}'\|_V \tag{3}$$

Eventually, from $\mathbf{S}\boldsymbol{\mu} = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1$ (resp. $\mathbf{S}\boldsymbol{\mu}' = \boldsymbol{\lambda}'_0 + \boldsymbol{\lambda}'_1$ ) it follows that

$$\|\mathbf{S}\boldsymbol{\mu} - \mathbf{S}\boldsymbol{\mu}'\|_V \leq \|\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}'_0\|_V + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}'_1\|_V \tag{4}$$

Finally, by combining (1), (2), (3) and (4) we obtain

$$\|\mathbf{T}\boldsymbol{\mu} - \mathbf{T}\boldsymbol{\mu}'\|_M \leq k\|\boldsymbol{\mu} - \boldsymbol{\mu}'\|_M$$

with $k = \frac{c_a c_c}{\beta^2}(1 + \frac{c_a}{\alpha_a})$, which ends the proof of theorem 4 since the operator $\mathbf{T}$ is a strict contraction as soon as the positive constant $k$ satisfies $k < 1$.

## 4. APPLICATION TO GEOMECHANICAL BOUNDARY VALUE PROBLEMS

This section deals with geomechanical boundary value problems coming from the analysis of quasistatic large transformations (i.e. large displacements, large strains and large rotations) of elastoviscoplastic materials. The space and time continuous problems considered are first described. A rate-type weak formulation of the space continuous problems coming from their time discretization is then performed. Eventually, theorem 4 is used for discussing the existence and uniqueness of the solution of these variational problems.

*4.1. Statement of the problem*

Let $\Omega$ be a materially simple continuum with elastoviscoplastic behaviour, the motion of which is studied over the time interval $[0, \mathrm{T}]$. We denote by $\Omega_t$ the configuration of $\Omega$ relating to the time $t \in [0, \mathrm{T}]$. We assume that $\Omega_t$ is an open, bounded and simply-connected region of the physical space $\mathbb{R}^3$, with Lipschitz-continuous boundary $\Gamma_t$, and we denote by $\mathbf{n}_t$ the outer unit normal to $\Gamma_t$. The successive configurations of $\Omega$ are observed with respect to the same fixed orthonormal frame, the time $t = 0$ being chosen as the reference value. The positions of the material particles of $\Omega$ are given by the vectors $X \in \Omega_0$ at time $t = 0$, and by the vectors $x \in \Omega_t$ at the current time $t$. The transformation $\mathcal{F}_t : X \to x$ relating to that time is then a one-to-one mapping from $\Omega_0$ onto $\Omega_t$.

Let $\mathbf{b}(t, .) : \Omega_t \to \mathbb{R}^3$ be the vector field at time $t$ of the body forces acting per mass unit in $\Omega_t$ and let $\rho(t, .) : \Omega_t \to \mathbb{R}$ be the field of mass density relating to that time. We denote by $\Gamma_{t1}$ the part of $\Gamma_t$ on which we have, at time $t$, the essential boundary conditions $\mathbf{u}|\Gamma_{t1} = \mathbf{u}^1$, where $\mathbf{u}^1(t, .) : \Gamma_{t1} \to \mathbb{R}^3$ are the values of the displacement $\mathbf{u}(t, .)$ given on $\Gamma_{t1}$, and we denote by $\Gamma_{t2}$ the part of $\Gamma_t$ on which the values of the stress vector are prescribed at the same time. We assume that $\Gamma_{t1}$ and $\Gamma_{t2}$ constitute, at every time $t$, a partition of $\Gamma_t$ such that $\Gamma_{t1}$ has at least three non-aligned points, and we denote by $\mathbf{g}(t, .) : \Gamma_{t2} \to \mathbb{R}^3$ the values of the stress vector given on $\Gamma_{t2}$. Eventually, we consider only quasistatic problems for which the acceleration may be ignored.

Let then $\mathbf{v}(t, .) : \Omega_t \to \mathbb{R}^3$ be the velocity field of the material body $\Omega$ at time $t$. We denote by $\boldsymbol{\varepsilon}(\mathbf{v})$ the strain rate tensor, symmetric part of the velocity gradient $\mathbf{grad}_x(\mathbf{v})$, and by $\boldsymbol{\omega}(\mathbf{v})$ the spin tensor, that is to say the skew-symmetric part of $\mathbf{grad}_x(\mathbf{v})$. We assume that the behaviour of the continuum $\Omega$ is governed by constitutive relations taking the form

$$\boldsymbol{\varepsilon}(\mathbf{v}) \quad = \quad \mathbf{H}\left(\check{\boldsymbol{\sigma}}, \mathcal{H}\right) + \mathbf{L}\left(\mathcal{H}\right) \tag{5}$$

where $\check{\boldsymbol{\sigma}}$ is the Jaumann's objective rate of the Cauchy's stress tensor $\boldsymbol{\sigma}$, $\mathcal{H}$ the set of memory parameters at the material point considered and where the $\mathcal{H}$-dependent tensor $\mathbf{L}$ represents the viscous part of $\boldsymbol{\varepsilon}(\mathbf{v})$, whereas $\mathbf{H}$ is an $\mathcal{H}$-dependent tensorial function of $\check{\boldsymbol{\sigma}}$, positively homogeneous of degree one with respect to $\check{\boldsymbol{\sigma}}$ since it describes the rate-independent (i.e. non-viscous) part of $\boldsymbol{\varepsilon}(\mathbf{v})$. This behaviour's non-viscous component is hypoelastic if $\mathbf{H}$ is linear with respect to $\check{\boldsymbol{\sigma}}$ and anelastic otherwise. In that last case, by setting $\mathbf{L} = \mathbf{0}$ we obtain the class of classical elastoplastic relations expressing the strain rate tensor $\boldsymbol{\varepsilon}(\mathbf{v})$ as a function of the Jaumann's objective derivative of the Cauchy's stresses, but also the set of rate-independent laws based upon the same principle of expression of $\boldsymbol{\varepsilon}(\mathbf{v})$ and called "incremental laws involving interpolations", which in particular differ from the previous ones on account of the absence of an elastic component. In other words, the resulting constitutive equations are irreversible even for very small stress levels. The use of such laws for describing the non-linear behaviour of geomaterials such as soils is well established and many rheological models have been issued from this formalism[7][5][13].

The aim of the following section is to build a sequence of time discretized mixed variational problems involving both fields of velocity and objective Jaumann's stress rate, which avoids the inversion of the constitutive equations (5) while dissociating the time integration of the various fields from the rate-type finite element computations.

*4.2. Weak formulation of the time discretized problem*

Let $t_0, t_1, \dots, t_N,\ N \in \mathbb{N}^*$, be an increasing sequence of time values such that $t_0 = 0$ and $t_N = \mathrm{T}$. We assume that these values are chosen in such a way that the partition of $\Gamma_0$ defined by $\mathcal{F}_t^{-1}(\Gamma_{t1})$ and $\mathcal{F}_t^{-1}(\Gamma_{t2})$ remains time independent over each of the intervals $[t_{n-1}, t_n[,\ n \in \{1, \dots, N\}$. Since our double aim is to avoid the inversion of the constitutive equations (5) and dissociate the time integration of the various fields from the rate-type finite element computations, we take as unknowns of the problem the two fields appearing in these equations, that is to say the velocity $\mathbf{v}$ and the Jaumann's objective derivative $\check{\boldsymbol{\sigma}}$ of the Cauchy's stresses[14]. We are then interested, for $n \in \{0, \dots, N-1\}$, in finding the fields $\mathbf{v}$ and $\check{\boldsymbol{\sigma}}$ relating to the time value $t_n$. After solving this problem and independently of it, the time integration of the various fields is carried out so as to allow the resolution of the problem relating to time $t_{n+1}$. In all the following we shall put $\mathbf{v}_n(x) = \mathbf{v}(t_n, x)$, $\forall n \in \{0, \dots, N\}$ and $\forall x \in \Omega_{t_n}$, as well as analogous notations for $\boldsymbol{\sigma}$, $\check{\boldsymbol{\sigma}}$, $\mathcal{H}$, $\mathbf{b}$, $\mathbf{g}$, $\rho$ and for the other fields which will be introduced subsequently. Eventually, we denote by $L^2(\Omega_{t_n})$ the space of square integrable real functions defined on $\Omega_{t_n}$ and by $H^1(\Omega_{t_n})$ the Sobolev space of square integrable real functions defined on $\Omega_{t_n}$ with square integrable generalized derivatives of order one.

Let then $n \in \{0, \dots, N-1\}$, $V_n = \left[L^2(\Omega_{t_n})\right]^9_{\mathrm{sym}}$, $M_n = \left[H^1(\Omega_{t_n})\right]^3$, and let $M_{0n}$ denote the closed subspace of $M_n$ constituted by the fields $\mathbf{v}$ such that $\mathbf{v}|\Gamma_{t_n 1} = 0$. We obtain the first part of the mixed variational formulation of the time discretized problem relating to time $t_n$ by doing the inner product of $\mathbf{s} \in V_n$ with the constitutive equations (5) written at that time, and then by integrating the resulting expression on $\Omega_{t_n}$. So we have

$$\int_{\Omega_{t_n}} \mathbf{H}(\check{\boldsymbol{\sigma}}_n, \mathcal{H}_n) : \mathbf{s}\, \mathrm{d}\Omega_{t_n} + \int_{\Omega_{t_n}} \mathbf{L}(\mathcal{H}_n) : \mathbf{s}\, \mathrm{d}\Omega_{t_n} \quad = \quad \int_{\Omega_{t_n}} \mathbf{s} : \boldsymbol{\varepsilon}(\mathbf{v}_n)\, \mathrm{d}\Omega_{t_n}$$

As to the second part of that formulation, it is obtained by considering an objective time derivative of the equations coming from the balance principle of linear momentum. The inner product of this derivative with $\mathbf{w} \in M_{0n}$ is first performed, before integrating it on $\Omega_{t_n}$. After integration by parts and use of the Gauss integral identity we obtain, independently of the objective derivative considered[6][15], the following relation

$$\begin{aligned} \int_{\Omega_{t_n}} \check{\boldsymbol{\sigma}}_n : \boldsymbol{\varepsilon}(\mathbf{w})\, \mathrm{d}\Omega_{t_n} + \int_{\Omega_{t_n}} \left[\mathbf{grad}_x(\mathbf{v}_n).\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_n \mathrm{div}_x(\mathbf{v}_n) - \boldsymbol{\varepsilon}(\mathbf{v}_n).\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n.\boldsymbol{\varepsilon}(\mathbf{v}_n)\right] : \mathbf{grad}_x(\mathbf{w})\, \mathrm{d}\Omega_{t_n} \\ = \int_{\Omega_{t_n}} \rho_n \dot{\mathbf{b}}_n.\mathbf{w}\, \mathrm{d}\Omega_{t_n} + \int_{\Gamma_{t_n 2}} \left[\dot{\mathbf{g}}_n + \left[\mathrm{div}_x(\mathbf{v}_n) - \mathbf{n}_{t_n}.\boldsymbol{\varepsilon}(\mathbf{v}_n).\mathbf{n}_{t_n}\right] \mathbf{g}_n\right].\mathbf{w}\, \mathrm{d}\Gamma_{t_n} \end{aligned}$$

where the point above a given field represents the material derivation.

Let now $\mathbf{v}^1_n$ be the rate of the displacement given on $\Gamma_{t_n 1}$, $M^{\perp}_{0n}$ be the orthogonal set of $M_{0n}$ in $M_n$ and let $\mathbf{v}_{1n}$ be the unique element of $M^{\perp}_{0n}$ such that $\mathbf{v}_{1n}|\Gamma_{t_n 1} = \mathbf{v}^1_n$. Then $\mathbf{v}_n = \mathbf{v}_{0n} + \mathbf{v}_{1n}$ with $\mathbf{v}_{0n} \in M_{0n}$, so that the mixed variational problem $(P_n)$ relating to time $t_n$ takes the following abstract form[6][15]

$$(P_n) \left\{ \begin{array}{lllll} \multicolumn{5}{l}{\text{Find } (\check{\boldsymbol{\sigma}}_n, \mathbf{v}_{0n}) \in V_n \times M_{0n} \text{ such that:}} \\ a_n(\check{\boldsymbol{\sigma}}_n, \mathbf{s}) & - \; b_n(\mathbf{s}, \mathbf{v}_{0n}) & = & h_n(\mathbf{s}) & \forall \mathbf{s} \in V_n \\ b_n(\check{\boldsymbol{\sigma}}_n, \mathbf{w}) & + \; c_n(\mathbf{v}_{0n}, \mathbf{w}) & = & l_n(\mathbf{w}) & \forall \mathbf{w} \in M_{0n} \end{array} \right. \qquad (6)$$

with

$$a_n(\check{\boldsymbol{\sigma}}_n, \mathbf{s}) = \int_{\Omega_{t_n}} \mathbf{H}\,(\check{\boldsymbol{\sigma}}_n, \mathcal{H}_n) : \mathbf{s}\, \mathrm{d}\Omega_{t_n} \tag{7}$$

$$b_n(\mathbf{s}, \mathbf{v}_{0n}) = \int_{\Omega_{t_n}} \mathbf{s} : \boldsymbol{\varepsilon}(\mathbf{v}_{0n})\, \mathrm{d}\Omega_{t_n} \tag{8}$$

$$b_n(\check{\boldsymbol{\sigma}}_n, \mathbf{w}) = \int_{\Omega_{t_n}} \check{\boldsymbol{\sigma}}_n : \boldsymbol{\varepsilon}(\mathbf{w})\, \mathrm{d}\Omega_{t_n} \tag{9}$$

$$\begin{aligned} c_n(\mathbf{v}_{0n}, \mathbf{w}) = & \int_{\Omega_{t_n}} [\mathbf{grad}_x(\mathbf{v}_{0n}).\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_n \mathrm{div}_x(\mathbf{v}_{0n}) - \boldsymbol{\varepsilon}(\mathbf{v}_{0n}).\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n.\boldsymbol{\varepsilon}(\mathbf{v}_{0n})] : \mathbf{grad}_x(\mathbf{w})\, \mathrm{d}\Omega_{t_n} \\ & - \int_{\Gamma_{t_n 2}} [\mathrm{div}_x(\mathbf{v}_{0n}) - \mathbf{n}_{t_n}.\boldsymbol{\varepsilon}(\mathbf{v}_{0n}).\mathbf{n}_{t_n}]\, \mathbf{g}_n.\mathbf{w}\, \mathrm{d}\Gamma_{t_n} \end{aligned} \tag{10}$$

and

$$h_n(\mathbf{s}) = \int_{\Omega_{t_n}} -\mathbf{L}\,(\mathcal{H}_n) : \mathbf{s}\, \mathrm{d}\Omega_{t_n} + b_n(\mathbf{s}, \mathbf{v}_{1n}) \tag{11}$$

$$l_n(\mathbf{w}) = \int_{\Omega_{t_n}} \rho_n \dot{\mathbf{b}}_n.\mathbf{w}\, \mathrm{d}\Omega_{t_n} + \int_{\Gamma_{t_n 2}} \dot{\mathbf{g}}_n.\mathbf{w}\, \mathrm{d}\Gamma_{t_n} - c_n(\mathbf{v}_{1n}, \mathbf{w}) \tag{12}$$

### *4.3. Discussion*

In this section we purpose to use theorem 4 in order to discuss the existence and the uniqueness of the solution of problem $(P_n)$.

Note first that $(V_n)$ and $(M_n)$ are Hilbert spaces when equipped with the inner products

$$(\boldsymbol{\sigma}, \mathbf{s})_{V_n} = \int_{\Omega_{t_n}} \boldsymbol{\sigma} : \mathbf{s}\, \mathrm{d}\Omega_{t_n} \qquad \forall (\boldsymbol{\sigma}, \mathbf{s}) \in V_n \times V_n \tag{13}$$

and

$$(\mathbf{v}, \mathbf{w})_{M_n} = \int_{\Omega_{t_n}} [\mathbf{v}.\mathbf{w} + \mathbf{grad}_x(\mathbf{v}) : \mathbf{grad}_x(\mathbf{w})]\, \mathrm{d}\Omega_{t_n} \qquad \forall (\mathbf{v}, \mathbf{w}) \in M_n \times M_n \tag{14}$$

respectively.

Thus, the closed subspace $M_{0n}$ of $M_n$ is also an Hilbert space when equipped with the inner product of $M_n$. Otherwise we have, from (13) and (14)

$$\|\mathbf{w}\|^2_{M_n} = \|\mathbf{w}\|^2_{[L^2(\Omega_{t_n})]^3} + \|\mathbf{grad}_x(\mathbf{w})\|^2_{V_n} \qquad \forall \mathbf{w} \in M_n$$

which gives

$$\|\mathbf{w}\|_{[L^2(\Omega_{t_n})]^3} \leq \|\mathbf{w}\|_{M_n} \qquad \forall \mathbf{w} \in M_n \tag{15}$$

and

$$\|\mathbf{grad}_x(\mathbf{w})\|_{V_n} \leq \|\mathbf{w}\|_{M_n} \qquad \forall \mathbf{w} \in M_n \tag{16}$$

We have also, since $\boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\omega}(\mathbf{w}) = 0$, $\forall \mathbf{w} \in M_n$ and $\forall x \in \Omega_{t_n}$,

$$\|\mathbf{grad}_x(\mathbf{w})\|^2_{V_n} = \|\boldsymbol{\varepsilon}(\mathbf{w})\|^2_{V_n} + \|\boldsymbol{\omega}(\mathbf{w})\|^2_{V_n} \qquad \forall \mathbf{w} \in M_n$$

and then, taking into account (16),

$$\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{V_n} \leq \|\mathbf{w}\|_{M_n} \qquad \forall \mathbf{w} \in M_n \tag{17}$$

Eventually, from the decomposition $\boldsymbol{\varepsilon}(\mathbf{w}) = \frac{1}{3}\mathrm{div}_x(\mathbf{w})\boldsymbol{\delta} + \mathbf{e}(\mathbf{w})$ of $\boldsymbol{\varepsilon}(\mathbf{w})$ into isotropic and deviatoric parts it follows, since $\mathbf{e}(\mathbf{w}):\boldsymbol{\delta} = 0$, $\forall \mathbf{w} \in M_n$ and $\forall x \in \Omega_{t_n}$, that

$$\|\boldsymbol{\varepsilon}(\mathbf{w})\|^2_{V_n} = \frac{1}{3}\|\mathrm{div}_x(\mathbf{w})\|^2_{L^2(\Omega_{t_n})} + \|\mathbf{e}(\mathbf{w})\|^2_{V_n} \qquad \forall \mathbf{w} \in M_n$$

which gives, with (17)

$$\|\mathrm{div}_x(\mathbf{w})\|_{L^2(\Omega_{t_n})} \leq \sqrt{3}\|\mathbf{w}\|_{M_n} \qquad \forall \mathbf{w} \in M_n \tag{18}$$

Let us now consider the form $b_n : V_n \times M_n \mapsto \mathbb{R}$ defined by (8) or (9). This form is obviously bilinear. It is also continuous since we have, taking into account (17),

$$|b_n(\mathbf{s}, \mathbf{w})| = |\,(\mathbf{s}, \boldsymbol{\varepsilon}(\mathbf{w}))_{V_n}\,| \leq \|\mathbf{s}\|_{V_n}\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{V_n} \leq \|\mathbf{s}\|_{V_n}\|\mathbf{w}\|_{M_n} \qquad \forall (\mathbf{s}, \mathbf{w}) \in V_n \times M_n \tag{19}$$

Hence, the restriction of $b_n$ to $V_n \times M_{0n}$ is bilinear and continuous, with continuity constant $c_b = 1$

Let us then show that the restriction of $b_n$ to $V_n \times M_{0n}$ satisfies the inf-sup condition of lemma 1. Note first that $\Gamma_{t_n1}$ has at least three non-aligned points. Thus, because of the Korn's inequality there exists $\gamma \in \mathbb{R}^{+\star}$ such that

$$\|\boldsymbol{\varepsilon}(\mathbf{w})\|^2_{V_n} \geq \gamma\|\mathbf{w}\|^2_{M_n} \qquad \forall \mathbf{w} \in M_{0n} \tag{20}$$

which gives, $\forall \mathbf{w} \in \times M^\star_{0n}$, $\boldsymbol{\varepsilon}(\mathbf{w}) \in V^\star_n$ and

$$\sup_{\mathbf{s}\in V_n^\star} \frac{b(\mathbf{s}, \mathbf{w})}{\|\mathbf{s}\|_{V_n}\|\mathbf{w}\|_{M_n}} = \sup_{\mathbf{s}\in V_n^\star} \frac{(\mathbf{s}, \boldsymbol{\varepsilon}(\mathbf{w}))_{V_n}}{\|\mathbf{s}\|_{V_n}\|\mathbf{w}\|_{M_n}} \geq \frac{(\boldsymbol{\varepsilon}(\mathbf{w}), \boldsymbol{\varepsilon}(\mathbf{w}))_{V_n}}{\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{V_n}\|\mathbf{w}\|_{M_n}} = \frac{\|\boldsymbol{\varepsilon}(\mathbf{w})\|_{V_n}}{\|\mathbf{w}\|_{M_n}} \geq \sqrt{\gamma}$$

Thus, since $\|\mathbf{w}\|_{M_{0n}} = \|\mathbf{w}\|_{M_n}$ $\forall \mathbf{w} \in M_{0n}$,

$$\inf_{\mathbf{w}\in M^\star_{0n}} \sup_{\mathbf{s}\in V^\star_n} \frac{b(\mathbf{s}, \mathbf{w})}{\|\mathbf{s}\|_{V_n}\|\mathbf{w}\|_{M_{0n}}} \geq \sqrt{\gamma}$$

and the inf-sup condition of lemma 1 holds with $\beta = \sqrt{\gamma} > 0$.

We are now interested in the form $c_n : M_n \times M_n \mapsto \mathbb{R}$ given by (10). In order that the integrals defining that form may be defined, we assume that $\boldsymbol{\sigma}_n \in [L^\infty(\Omega_{t_n})]^9_{\mathrm{sym}}$ and $\mathbf{g}_n \in [L^\infty(\Gamma_{t_n2})]^3$. We have then

$$\|\boldsymbol{\sigma}_n\|_\infty = \inf\left\{c \in \mathbb{R}^+, \|\boldsymbol{\sigma}_n(x)\| \leq c \text{ for nearly all } x \in \Omega_{t_n}\right\}$$

and

$$\|\mathbf{g}_n\|_\infty = \inf\left\{c \in \mathbb{R}^+, \|\mathbf{g}_n(x)\| \leq c \text{ for nearly all } x \in \Gamma_{t_n2}\right\}$$

where $\|.\|$ denotes the euclidian norm.

Note first that like $b_n$ the form $c_n$ is bilinear. Moreover we have, $\forall (\mathbf{v}, \mathbf{w}) \in M_n \times M_n$,

$$\begin{aligned}|c_n(\mathbf{v}, \mathbf{w})| \leq\ & \|\mathbf{grad}_x(\mathbf{v}).\boldsymbol{\sigma}_n + \boldsymbol{\sigma}_n\mathrm{div}_x(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{v}).\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n.\boldsymbol{\varepsilon}(\mathbf{v})\|_{V_n}\|\mathbf{grad}_x(\mathbf{w})\|_{V_n} \\ & + \|\mathrm{div}_x(\mathbf{v}) - \mathbf{n}_{t_n}.\boldsymbol{\varepsilon}(\mathbf{v}).\mathbf{n}_{t_n}\|_{L^2(\Gamma_{t_n2})}\|\mathbf{g}_n.\mathbf{w}\|_{L^2(\Gamma_{t_n2})} \\ \leq\ & \left[\|\mathbf{grad}_x(\mathbf{v}).\boldsymbol{\sigma}_n\|_{V_n} + \|\boldsymbol{\sigma}_n\mathrm{div}_x(\mathbf{v})\|_{V_n} + \|\boldsymbol{\varepsilon}(\mathbf{v}).\boldsymbol{\sigma}_n\|_{V_n} + \|\boldsymbol{\sigma}_n.\boldsymbol{\varepsilon}(\mathbf{v})\|_{V_n}\right] \|\mathbf{grad}_x(\mathbf{w})\|_{V_n} \\ & + \|\mathrm{div}_x(\mathbf{v}) - \mathbf{n}_{t_n}.\boldsymbol{\varepsilon}(\mathbf{v}).\mathbf{n}_{t_n}\|_{L^2(\Gamma_{t_n2})}\|\mathbf{g}_n.\mathbf{w}\|_{L^2(\Gamma_{t_n2})}\end{aligned} \tag{21}$$

 

By denoting again as $\|.\|$ the euclidian norm we have also, $\forall \mathbf{s} \in V_n$, $\forall \mathbf{w} \in M_n$ and for nearly all $x \in \Omega_{t_n}$,

$$\|\mathbf{s}.\boldsymbol{\sigma}_n\| = \|\boldsymbol{\sigma}_n.\mathbf{s}\| \leq \|\boldsymbol{\sigma}_n\|_\infty \|\mathbf{s}\|$$

as well as

$$\|\boldsymbol{\sigma}_n \mathrm{div}_x(\mathbf{w})\| \leq \|\boldsymbol{\sigma}_n\|_\infty |\mathrm{div}_x(\mathbf{w})|$$

Likewise we have, $\forall \mathbf{w} \in M_n$ and for nearly all $x \in \Gamma_{t_n 2}$,

$$|\mathbf{g}_n.\mathbf{w}| \leq \|\mathbf{g}_n\|_\infty \|\mathbf{w}\|$$

Thus we obtain, $\forall \mathbf{s} \in V_n$ and $\forall \mathbf{w} \in M_n$,

$$\|\mathbf{s}.\boldsymbol{\sigma}_n\|_{V_n} = \|\boldsymbol{\sigma}_n.\mathbf{s}\|_{V_n} \leq \|\boldsymbol{\sigma}_n\|_\infty \|\mathbf{s}\|_{V_n}$$

as well as

$$\|\boldsymbol{\sigma}_n \mathrm{div}_x(\mathbf{w})\|_{V_n} \leq \|\boldsymbol{\sigma}_n\|_\infty \|\mathrm{div}_x(\mathbf{w})\|_{L^2(\Omega_{t_n})}$$

and

$$\|\mathbf{g}_n.\mathbf{w}\|_{L^2(\Gamma_{t_n 2})} \leq \|\mathbf{g}_n\|_\infty \|\mathbf{w}\|_{[L^2(\Gamma_{t_n 2})]^3}$$

so that the inequality (21) becomes

$$\begin{aligned} |c_n(\mathbf{v},\mathbf{w})| \leq \|\boldsymbol{\sigma}_n\|_\infty \left[\|\mathbf{grad}_x(\mathbf{v})\|_{V_n} + \|\mathrm{div}_x(\mathbf{v})\|_{L^2(\Omega_{t_n})} + 2\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{V_n}\right] \|\mathbf{grad}_x(\mathbf{w})\|_{V_n} \\ + \|\mathbf{g}_n\|_\infty \|\mathrm{div}_x(\mathbf{v}) - \mathbf{n}_{t_n}.\boldsymbol{\varepsilon}(\mathbf{v}).\mathbf{n}_{t_n}\|_{L^2(\Gamma_{t_n 2})} \|\mathbf{w}\|_{[L^2(\Gamma_{t_n 2})]^3} \end{aligned} \quad (22)$$

Moreover, from the trace theorem there exists $c_t \in \mathbb{R}^{+\star}$ such that, $\forall \mathbf{w} \in M_n$,

$$\|\mathbf{w}\|_{[L^2(\Gamma_{t_n 2})]^3} \leq c_t \|\mathbf{w}\|_{M_n} \quad (23)$$

as well as

$$\|\mathrm{div}_x(\mathbf{w}) - \mathbf{n}_{t_n}.\boldsymbol{\varepsilon}(\mathbf{w}).\mathbf{n}_{t_n}\|_{L^2(\Gamma_{t_n 2})} \leq c_t \|\mathbf{w}\|_{M_n}$$

Thus we obtain finally, taking into account (16), (17), (18) and (22),

$$|c_n(\mathbf{v},\mathbf{w})| \leq c_c \|\mathbf{v}\|_{M_n} \|\mathbf{w}\|_{M_n} \qquad \forall (\mathbf{v},\mathbf{w}) \in M_n \times M_n \quad (24)$$

with

$$c_c = (3+\sqrt{3})\|\boldsymbol{\sigma}_n\|_\infty + c_t^2 \|\mathbf{g}_n\|_\infty \quad (25)$$

which ensures the continuity of $c_n$ as well as the continuity of its restriction to $M_{0n} \times M_{0n}$.

Let us now consider the forms $h_n : V_n \mapsto \mathbb{R}$ and $l_n : M_n \mapsto \mathbb{R}$ given by (11) and (12), respectively. These forms are obviously linear and we have, taking into account (15), (19), (23) and (24)

$$\begin{aligned} |h_n(\mathbf{s})| &\leq \|\mathbf{L}(\mathcal{H}_n)\|_{V_n} \|\mathbf{s}\|_{V_n} + |b_n(\mathbf{s},\mathbf{v}_{1n})| \\ &\leq (\|\mathbf{L}(\mathcal{H}_n)\|_{V_n} + \|\mathbf{v}_{1n}\|_{M_n}) \|\mathbf{s}\|_{V_n} \end{aligned}$$

as well as

$$\begin{aligned} |l_n(\mathbf{w})| &\leq \|\rho_n \dot{\mathbf{b}}_n\|_{[L^2(\Omega_{t_n})]^3} \|\mathbf{w}\|_{[L^2(\Omega_{t_n})]^3} + \|\dot{\mathbf{g}}_n\|_{[L^2(\Gamma_{t_n 2})]^3} \|\mathbf{w}\|_{[L^2(\Gamma_{t_n 2})]^3} + |c_n(\mathbf{v}_{1n},\mathbf{w})| \\ &\leq \left(\|\rho_n \dot{\mathbf{b}}_n\|_{[L^2(\Omega_{t_n})]^3} + c_t \|\dot{\mathbf{g}}_n\|_{[L^2(\Gamma_{t_n 2})]^3} + c_c \|\mathbf{v}_{1n}\|_{M_n}\right) \|\mathbf{w}\|_{M_n} \end{aligned}$$

Hence, $h_n \in V_n'$ and $l_n \in M_n' \supset M_{n0}'$.

Let us now study the form $a_n : M_n \times M_n \mapsto \mathbb{R}$ defined by (7). We have

$$a_n(\check{\boldsymbol{\sigma}}_n, \mathbf{s}) = \left(\mathbf{H}\left(\check{\boldsymbol{\sigma}}_n, \mathcal{H}_n\right), \mathbf{s}\right)_{V_n}$$

so that this form is linear and continuous with respect to its second argument. Moreover, the operator $\mathbf{A}$ of theorem 4 coincides with the rheological operator $\mathbf{H}$. Thus, the conditions 1 and 2 of theorem 4 hold if and only if they are satisfied by this operator. However, they remain difficult to state except for the case where $\mathbf{H}$ is linear with respect to $\check{\boldsymbol{\sigma}}_n$. And indeed, the difficulty lies in the fact that conditions 1 and 2 of theorem 4 are global (in the sense that they are expressed by the way of integrals on $\Omega_{t_n}$) whereas the constitutive relations (5) are defined locally (i.e. at each point $x$ of $\Omega_{t_n}$). One can nevertheless try to provide regularity conditions of these constitutive equations sufficient to ensure that conditions 1 and 2 of theorem 4 hold.

An indeed, let us denote by $\|.\|$ the euclidian norm and assume that there exists $c_H \in \mathbb{R}^{+\star}$ and $\alpha_H \in \mathbb{R}^{+\star}$ such that we have, $\forall(\check{\boldsymbol{\sigma}}_n, \check{\boldsymbol{\sigma}}_n') \in V_n \times V_n$ and for nearly all $x \in \Omega_{t_n}$,

$$\|\mathbf{H}\left(\check{\boldsymbol{\sigma}}_n(x), \mathcal{H}_n(x)\right) - \mathbf{H}\left(\check{\boldsymbol{\sigma}}_n'(x), \mathcal{H}_n(x)\right)\| \quad \leq \quad c_H \|\check{\boldsymbol{\sigma}}_n(x), -\check{\boldsymbol{\sigma}}_n'(x)\| \quad (26)$$

$$\left(\mathbf{H}\left(\check{\boldsymbol{\sigma}}_n(x), \mathcal{H}_n(x)\right) - \mathbf{H}\left(\check{\boldsymbol{\sigma}}_n'(x), \mathcal{H}_n(x)\right)\right) : \left(\check{\boldsymbol{\sigma}}_n(x), -\check{\boldsymbol{\sigma}}_n'(x)\right) \quad \geq \quad \alpha_H \|\check{\boldsymbol{\sigma}}_n(x), -\check{\boldsymbol{\sigma}}_n'(x)\|^2 \quad (27)$$

Thus, conditions 1 and 2 of theorem 4 hold with $c_a = c_H$ and $\alpha_a = \alpha_H$.

However, the relations (26) and (27) are sufficient but non necessary conditions since the global conditions 1 and 2 of theorem 4 can be satisfied even if relations (26) and/or (27) do not hold.

The last point of the discussion deals with the condition

$$\frac{c_a c_c}{\beta^2}\left(1 + \frac{c_a}{\alpha_a}\right) < 1 \qquad (28)$$

of theorem 4. We have here $\beta^2 = \gamma$ where $\gamma > 0$ is the constant of relation (20) coming from the Korn's inequality, so that condition (28) can be also written as

$$c_a c_c < \frac{\gamma}{1 + \frac{c_a}{\alpha_a}} \qquad (29)$$

Since we have always $\gamma < 1$ and $\alpha_a \leq c_a$, the inequality (29) hold if and only if the product $c_a c_c$ is sufficiently less than 1. Now the expression (25) of $c_c$ shows that the magnitude of this strictly positive constant is linked to that of the state of stress $\boldsymbol{\sigma}_n$. Likewise, the coincidence between the operator $\mathbf{A}$ of theorem 4 and the rheological operator $\mathbf{H}$ together with the condition 1 of this theorem show that the magnitude of $c_a$ is in relation with that of the components of the tangent matrices defining $\mathbf{H}$ at each point $x$ of $\Omega_{t_n}$, that is to say with the magnitude of the inverses of the tangent Young's moduli $E$ at each point $x$ of $\Omega_{t_n}$. Hence, since constants $c_a$ and $c_c$ are defined globally (i.e. by the way of integrals on $\Omega_{t_n}$), the product $c_a c_c$ can be sufficiently less than 1 if the ratio $\frac{\|\boldsymbol{\sigma}_n\|}{E}$ remains sufficiently small for nearly all $x \in \Omega_{t_n}$.

## 5. CONCLUSION

In this paper we established some existence and uniqueness results for the solution of variational problems constituting the basis of finite element approximations encountered in mechanics and civil engineering. By expanding to the approximate problems coming from the space discretization, such theoretical results contribute to strengthen the robustness of the modelling softwares and the quality of their numerical results.

However, some of the conditions constituting the basis of these results, necessary and sufficient when linear constitutive equations are used, become difficult to state and lose their necessary feature when considering non-linear constitutive relations. And indeed, for such rheological models they can only provide regularity conditions of the constitutive equations which are sufficient but non necessary to ensure the existence and uniqueness of the solution of the variational problems considered. So, it is essential to study this last point thoroughly if one wants to obtain sufficient regularity conditions which are also necessary to the well-posedness of these problems.

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